SOME INTERSECTIONS OF LORENTZ SPACES

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Abstract. Let (X,μ) be a measure space. For $p,q \in (0,\infty]$ and arbitrary subsets P,Q of $(0,\infty]$, we introduce and characterize some intersections of Lorentz spaces, denoted by $IL_{p,Q}(X,\mu)$, $IL_{J,q}(X,\mu)$ and $IL_{J,Q}(X,\mu)$.

0. Introduction

Let (X,μ) be a measure space. For $0 < p \leq \infty$, the space $L^p(X,\mu)$ is the usual Lebesgue space as defined in [3] and [6]. Let us remark that for $1 \leq p < \infty$

$$\|f\|_p := \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p}$$

defines a norm on $L^p(X,\mu)$ such that $(L^p(X,\mu), \|.\|_p)$ is a Banach space. Also for $0 < p < 1$,

$$\|f\|_p := \int_X |f(x)|^p d\mu(x)$$

defines a quasi norm on $L^p(X,\mu)$ such that $(L^p(X,\mu), \|.\|_p)$ is a complete metric space. Moreover for $p = \infty$,

$$\|f\|_\infty = \inf\{B > 0 : \mu(\{x \in X : |f(x)| > B\}) = 0\}$$

defines a norm on $L^\infty(X,\mu)$ such that $(L^\infty(X,\mu), \|.\|_\infty)$ is a Banach space. In [1], we considered an arbitrary intersection of the L^p-spaces denoted by $\bigcap_{p\in J} L^p(G)$, where G is a locally compact group with a left Haar measure λ and $J \subseteq [1,\infty]$. Then we introduced the subspace $IL_J(G)$ of $\bigcap_{p\in J} L^p(G)$ as

$$IL_J(G) = \{f \in \bigcap_{p\in J} L^p(G) : \|f\|_J = \sup_{p\in J} \|f\|_p < \infty\},$$

and studied $IL_J(G)$ as a Banach algebra under convolution product, for the case where $1 \in J$. Also in [2], we generalized the results of [1] to the weighted case. In fact for an arbitrary family Ω of the weight functions on G and $1 \leq p < \infty$, we introduced the subspace $IL_p(G,\Omega)$ of the locally convex space $L^p(G,\Omega) = \bigcap_{\omega\in\Omega} L^p(G,\omega)$. Moreover, we provided some sufficient conditions on G and also Ω to construct a norm on $IL_p(G,\Omega)$. The fourth section of [2] has been assigned to some intersections of Lorentz spaces. Indeed for the case where p is fixed and q runs through $J \subseteq (0,\infty)$, we introduced $IL_{p,J}(G)$ as a subspace of $\cap_{q\in J} L_{p,q}(G)$, where $L_{p,q}(G)$ is the Lorentz space with indices p and q. As the main result, we proved that $IL_{p,J}(G) = L_{p,m_J}(G)$, in the case where $m_J = \inf\{q : q \in J\}$ is positive.

In the present work, we continue our study concerning the intersections of Lorentz spaces on the measure space (X,μ), to complete our results in this direction. Precisely, we verify most the results given in the second and third sections of [1], for Lorentz spaces.

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1. **Preliminaries**

In this section, we give some preliminaries and definitions which will be used throughout the paper. We refer to [3], as a good introductory book.

Let (X,μ) be a measure space and f be a complex valued measurable function on X. For each $\alpha>0$, let

$$d_f(\alpha)=\mu(\{x\in X:|f(x)|>\alpha\}).$$

The decreasing rearrangement of f is the function $f^*:[0,\infty)\to[0,\infty]$ defined by

$$f^*(t)=\inf\{s>0:d_f(s)\leq t\}.$$

We adopt the convention $\inf\emptyset=\infty$, thus having $f^*(t)=\infty$ whenever $d_f(\alpha)>t$ for all $\alpha\geq 0$. For $0<p\leq\infty$ and $0<q<\infty$, define

$$\|f\|_{L_{p,q}}=\left(\int_0^\infty\left(t^{\frac{1}{p}}f^*(t)\right)^q\frac{dt}{t}\right)^{1/q}, \tag{1.1}$$

where dt is the Lebesgue measure. In the case where $q=\infty$, define

$$\|f\|_{L_{p,\infty}}=\sup_{t>0}t^{\frac{1}{p}}f^*(t). \tag{1.2}$$

The set of all f with $\|f\|_{p,q}<\infty$ is denoted by $L_{p,q}(X,\mu)$ and is called the Lorentz space with indices p and q. As in L^p-spaces, two functions in $L_{p,q}(X,\mu)$ are considered equal if they are equal $\mu-$almost everywhere on X. It is worth noting that by [3, Proposition 1.4.5] for each $0<p<\infty$ we have

$$\int_X|f(x)|^p d\mu(x)=\int_0^\infty f^*(t)^p dt. \tag{1.3}$$

It follows that $L_{p,p}(X,\mu)=L^p(X,\mu)$. Furthermore by the definition given in equation (1.2), one can observe that $L_{\infty,\infty}(X,\mu)=L^\infty(X,\mu)$. Note that in the case where $p=\infty$, one can conclude that the only simple function with finite $\|.\|_{L_{\infty,q}}$ norm is the zero function. For this reason, $L_{\infty,q}(X,\mu)=\{0\}$, for every $0<q<\infty$; see [3, page 49].

In [2], for locally compact group G and $0<p<\infty$ and also an arbitrary subset Q of $(0,\infty)$ with

$$m_Q=\inf\{q:q\in Q\}>0,$$

we introduced $IL_{p,Q}(G)$ as a subset of $\cap_{q\in Q}L_{p,q}(G)$ by

$$IL_{p,Q}(G)=\{f\in\bigcap_{q\in Q}L_{p,q}(G):\|f\|_{L_{p,Q}}=\sup_{q\in Q}\|f\|_{L_{p,q}}<\infty\}. \tag{1.4}$$

As the main result of the third section in [2], we proved the following theorem.

Theorem 1.1. [2, Theorem 12] *Let G be a locally compact group, $0<p<\infty$ and Q be an arbitrary subset of $(0,\infty)$ such that $m_Q>0$. Then $IL_{p,Q}(G)=L_{p,m_Q}(G)$. Moreover, for each $f\in L_{p,m_Q}(G)$,*

$$\|f\|_{L_{p,m_Q}}\leq\|f\|_{L_{p,Q}}\leq\max\left\{1,\left(\frac{m_Q}{p}\right)^{1/m_Q}\right\}\|f\|_{L_{p,m_Q}}. \tag{1.5}$$

Note that in the definition of $IL_{p,Q}(G)$ given in (1.4), one can replace G by an arbitrary measure space (X,μ). Precisely if let

$$IL_{p,Q}(X,\mu)=\{f\in\bigcap_{q\in Q}L_{p,q}(X,\mu):\|f\|_{L_{p,Q}}=\sup_{q\in Q}\|f\|_{L_{p,q}}<\infty\}, \tag{1.6}$$

then $IL_{p,Q}(X,\mu) = L_{p,m_Q}(X,\mu)$. Moreover for each $f \in L_{p,m_Q}(X,\mu)$, (1.5) is satisfied. Furthermore, Theorem [2, Theorem 12] is also valid for $IL_{p,Q}(X,\mu)$. In the present work, in a similar way, we introduce and characterize the spaces $IL_{J,q}(X,\mu)$ and also $IL_{J,Q}(X,\mu)$, as other intersections of Lorentz spaces. Moreover we obtain some results about Lorentz space related to a Banach spaces E, introduced in [4].

2. **Main Results**

At the beginning of the present section we recall [3, Exercise 1.4.2], which will be used several times in our further arguments. A simple proof is given here.

Proposition 2.1. *Let (X,μ) be a measure space and $0 < p_1 < p_2 \leq \infty$. Then*

$$L_{p_1,\infty}(X,\mu) \bigcap L_{p_2,\infty}(X,\mu) \subseteq \bigcap_{p_1<p<p_2,0<s\leq\infty} L_{p,s}(X,\mu).$$

Proof. Let $f \in L_{p_1,\infty}(X,\mu) \cap L_{p_2,\infty}(X,\mu)$. If $\|f\|_{L_{p_1,\infty}} = 0$, one can readily obtained that $f \in L_{p,s}(X,\mu)$, for all $p_1 < p < p_2$ and $0 < s \leq \infty$. Now let $\|f\|_{L_{p_1,\infty}} \neq 0$ and first suppose that $p_2 < \infty$. We show that $f \in L_{p,s}(X,\mu)$, for all $p_1 < p < p_2$ and $0 < s < \infty$. It is clear that for each $\alpha > 0$

$$d_f(\alpha) \leq \min\left(\frac{\|f\|^{p_1}_{L_{p_1,\infty}}}{\alpha^{p_1}}, \frac{\|f\|^{p_2}_{L_{p_2,\infty}}}{\alpha^{p_2}}\right). \tag{2.1}$$

Set

$$B = \left(\frac{\|f\|^{p_2}_{L_{p_2,\infty}}}{\|f\|^{p_1}_{L_{p_1,\infty}}}\right)^{\frac{1}{p_2-p_1}}.$$

Thus

$$\begin{aligned}
\|f\|_{L_{p,s}} &= \left(p\int_0^\infty (d_f(\alpha)^{\frac{1}{p}}\ \alpha)^s \frac{d\alpha}{\alpha}\right)^{\frac{1}{s}} = \left(p\int_0^\infty d_f(\alpha)^{\frac{s}{p}}\ \alpha^{s-1} d\alpha\right)^{\frac{1}{s}} \\
&\leq \left(p\int_0^B \alpha^{s-1}\ \left(\frac{\|f\|^{p_1}_{L_{p_1,\infty}}}{\alpha^{p_1}}\right)^{\frac{s}{p}} d\alpha\right)^{\frac{1}{s}} + \left(p\int_B^\infty \alpha^{s-1}\ \left(\frac{\|f\|^{p_2}_{L_{p_2,\infty}}}{\alpha^{p_2}}\right)^{\frac{s}{p}} d\alpha\right)^{\frac{1}{s}} \\
&= p^{\frac{1}{s}}\ \|f\|^{\frac{p_1}{p}}_{L_{p_1,\infty}} \left(\int_0^B \alpha^{s-1-\frac{sp_1}{p}} d\alpha\right)^{\frac{1}{s}} + p^{\frac{1}{s}}\ \|f\|^{\frac{p_2}{p}}_{L_{p_2,\infty}} \left(\int_B^\infty \alpha^{s-1-\frac{sp_2}{p}} d\alpha\right)^{\frac{1}{s}} \\
&= p^{\frac{1}{s}}\ \|f\|^{\frac{p_1}{p}}_{L_{p_1,\infty}} \left(\frac{B^{s-\frac{sp_1}{p}}}{s-\frac{sp_1}{p}}\right)^{\frac{1}{s}} + p^{\frac{1}{s}}\ \|f\|^{\frac{p_2}{p}}_{L_{p_2,\infty}} \left(\frac{B^{s-\frac{sp_2}{p}}}{\frac{sp_2}{p}-s}\right)^{\frac{1}{s}} \\
&= \left(\frac{p}{(s-\frac{sp_1}{p})}\right)^{\frac{1}{s}}\ \|f\|^{\frac{p_1}{p}\cdot(\frac{p_2-p}{p_2-p_1})}_{L_{p_1,\infty}}\ \|f\|^{\frac{p_2}{p}\ (\frac{p-p_1}{p_2-p_1})}_{L_{p_2,\infty}} \\
&\quad + \left(\frac{p}{(\frac{sp_2}{p}-s)}\right)^{\frac{1}{s}}\ \|f\|^{\frac{p_1}{p}\cdot(\frac{p_2-p}{p_2-p_1})}_{L_{p_1,\infty}}\ \|f\|^{\frac{p_2}{p}\ (\frac{p-p_1}{p_2-p_1})}_{L_{p_2,\infty}} \\
&= \left(\left(\frac{p}{(s-\frac{sp_1}{p})}\right)^{\frac{1}{s}} + \left(\frac{p}{(\frac{sp_2}{p}-s)}\right)^{\frac{1}{s}}\right)\ \|f\|^{\frac{p_1}{p}(\frac{p_2-p}{p_2-p_1})}_{L_{p_1,\infty}}\ \|f\|^{\frac{p_2}{p}(\frac{p-p_1}{p_2-p_1})}_{L_{p_2,\infty}} \\
&< \infty.
\end{aligned}$$

Thus $f \in L_{p,s}(X,\mu)$. For $p_2 = \infty$, since $d_f(\alpha) = 0$ for each $\alpha > \|f\|_\infty$, inequality (2.1) implies that

$$\|f\|_{L_{p,s}}^s \leq \frac{p}{s - \frac{sp_1}{p}} \|f\|_{p_1,\infty}^{\frac{sp_1}{p}} \|f\|_\infty^{s-\frac{sp_1}{p}},$$

which implies $f \in L_{p,s}(X,\mu)$. In the case where $s = \infty$, by [3, Proposition 1.1.14], for $p_1 < r < p_2$ we have

$$L_{p_1,\infty}(X,\mu) \cap L_{p_2,\infty}(X,\mu) \subseteq L^r(X,\mu) \subseteq L_{r,\infty}(X,\mu).$$

This gives the proposition. □

Proposition 2.2. *Let (X,μ) be a measure space, $0 < q \leq \infty$ and $0 < p_1 < p_2 \leq \infty$. Then*

$$\bigcap_{p_1 \leq r \leq p_2} L_{r,q}(X,\mu) = L_{p_1,q}(X,\mu) \cap L_{p_2,q}(X,\mu).$$

Moreover for each $f \in L_{p_1,q}(X,\mu) \cap L_{p_2,q}(X,\mu)$ and $p_1 < r < p_2$,

$$\|f\|_{L_{r,q}} \leq 2^{1/q} \max\{\|f\|_{L_{p_1,q}}, \|f\|_{L_{p_2,q}}\}.$$

Proof. By [3, Proposition 1.4.10] and Proposition 2.1 we have

$$\begin{aligned} L_{p_1,q}(X,\mu) \cap L_{p_2,q}(X,\mu) &\subseteq L_{p_1,\infty}(X,\mu) \cap L_{p_2,\infty}(X,\mu) \\ &\subseteq \bigcap_{p_1<r<p_2, 0<s\leq\infty} L_{r,s}(X,\mu). \end{aligned}$$

It follows that

$$L_{p_1,q}(X,\mu) \cap L_{p_2,q}(X,\mu) \subseteq \bigcap_{p_1 \leq r \leq p_2} L_{r,q}(X,\mu).$$

The converse of the inclusion is clearly valid. Thus

$$L_{p_1,q}(X,\mu) \cap L_{p_2,q}(X,\mu) = \bigcap_{p_1 \leq r \leq p_2} L_{r,q}(X,\mu).$$

Now let $q < \infty$. For each $f \in L_{p_1,q}(X,\mu) \cap L_{p_2,q}(X,\mu)$, we have

$$\begin{aligned} \|f\|_{L_{r,q}}^q &= \int_0^\infty \left(t^{\frac{1}{r}} f^*(t)\right)^q \frac{dt}{t} \\ &\leq \int_0^1 \left(t^{\frac{1}{p_2}} f^*(t)\right)^q \frac{dt}{t} + \int_1^\infty \left(t^{\frac{1}{p_1}} f^*(t)\right)^q \frac{dt}{t} \\ &\leq \|f\|_{L_{p_2,q}}^q + \|f\|_{L_{p_1,q}}^q \\ &\leq 2 \max\{\|f\|_{L_{p_2,q}}^q, \|f\|_{L_{p_1,q}}^q\} \end{aligned}$$

Also for $q = \infty$ we have

$$\begin{aligned} \|f\|_{L_{r,\infty}} &= \sup_{t>0} t^{\frac{1}{r}} f^*(t) \\ &\leq \max\{\sup_{0<t<1} t^{\frac{1}{p_2}} f^*(t), \sup_{t\geq 1} t^{\frac{1}{p_1}} f^*(t)\} \\ &\leq \max\{\|f\|_{L_{p_2,\infty}}, \|f\|_{L_{p_1,\infty}}\}. \end{aligned}$$

This completes the proof. □

We are in a position to prove [1, Proposition 2.3] for Lorentz spaces. It is obtained in the following proposition. Recall from [1] that for a subset J of $(0,\infty)$,

$$M_J = \sup\{p : p \in J\}.$$

Proposition 2.3. *Let (X,μ) be a measure space, $0 < q \leq \infty$ and J be a subset of $(0,\infty)$ such that $0 < m_J$. Then the following assertions hold.*

(i) *If* $m_J, M_J \in J$, *then*
$$\bigcap_{p\in[m_J,M_J]} L_{p,q}(X,\mu) = \bigcap_{p\in J} L_{p,q}(X,\mu) = L_{m_J,q}(X,\mu)\cap L_{M_J,q}(X,\mu).$$
(ii) *If* $m_J \in J$ *and* $M_J \notin J$, *then* $\bigcap_{p\in J} L_{p,q}(X,\mu) = \bigcap_{p\in[m_J,M_J)} L_{p,q}(X,\mu)$.
(iii) *If* $m_J \notin J$ *and* $M_J \in J$, *then* $\bigcap_{p\in J} L_{p,q}(X,\mu) = \bigcap_{p\in(m_J,M_J]} L_{p,q}(X,\mu)$.
(iv) *If* $m_J, M_J \notin J$, *then* $\bigcap_{p\in J} L_{p,q}(X,\mu) = \bigcap_{p\in(m_J,M_J)} L_{p,q}(X,\mu)$.

Proof. (i). It is clearly obtain by Proposition 2.2.

(ii). Let $f\in\bigcap_{p\in J} L_{p,q}(X,\mu)$ and take $m_J < t < M_J$. Then there exist $t_1, t_2 \in J$ such that $t_1 < t < t_2$. So by Proposition 2.2
$$f\in L_{t_1,q}(X,\mu)\cap L_{t_2,q}(X,\mu) = \bigcap_{t_1\le p\le t_2} L_{p,q}(X,\mu)$$
and thus $f\in L_{t,q}(X,\mu)$. It follows that
$$\bigcap_{p\in J} L_{p,q}(X,\mu)\subseteq L_{t,q}(X,\mu),$$
for each $t\in[m_J,M_J)$. Consequently
$$\bigcap_{p\in J} L_{p,q}(X,\mu)\subseteq \bigcap_{p\in[m_J,M_J)} L_{p,q}(X,\mu).$$
The converse of the inclusion is clear.

(iii) and (iv) are proved in a similar way. □

Similar to the definition of $IL_{p,Q}(X,\mu)$ given in (1.6), for $J,Q\subseteq(0,\infty)$ let
$$IL_{J,q}(X,\mu) = \{f\in\textstyle\bigcap_{p\in J} L_{p,q}(X,\mu):\ \ \|f\|_{L_{J,q}} = \sup_{p\in J}\|f\|_{L_{p,q}} < \infty\}$$
and
$$IL_{J,Q}(X,\mu) = \{f\in\textstyle\bigcap_{p\in J,q\in Q} L_{p,q}(X,\mu):\ \ \|f\|_{L_{J,Q}} = \sup_{p\in J,q\in Q}\|f\|_{L_{p,q}} < \infty\}.$$

Proposition 2.4. *Let* (X,μ) *be a measure space,* $0<q\le\infty$ *and* $J\subseteq(0,\infty)$ *such that* $m_J>0$. *Then*
$$IL_{J,q}(X,\mu)\subseteq L_{m_J,q}(X,\mu)\cap L_{M_J,q}(X,\mu).$$

Proof. First, let $q<\infty$. We follow a proof similar to the proof of [2, Theorem 12]. Suppose that $M_J<\infty$ and (x_n) is a sequences in J such that $\lim_n x_n = M_J$. For $f\in IL_{J,q}(X,\mu)$ by Fatou's lemma, we have
$$\begin{aligned}
\|f\|^q_{L_{M_J,q}} &= \int_0^\infty \left(t^{\frac{1}{M_J}}.f^*(t)\right)^q \frac{dt}{t}\\
&= \int_0^\infty \liminf_n \left(t^{\frac{1}{x_n}}.f^*(t)\right)^q \frac{dt}{t}\\
&\le \liminf_n \int_0^\infty \left(t^{\frac{1}{x_n}}.f^*(t)\right)^q \frac{dt}{t}\\
&= \liminf_n \|f\|^q_{L_{x_n,q}}\\
&\le \|f\|^q_{J,q}\\
&< \infty.
\end{aligned}$$
If $M_J=\infty$ and $f\in IL_{J,q}(X,\mu)$, then
$$\begin{aligned}
\left(\int_0^\infty f^*(t)^q\frac{dt}{t}\right)^{1/q} &= \left(\int_0^\infty \liminf_n \left(t^{\frac{1}{x_n}} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q}\\
&\le \liminf_n \|f\|_{L^{x_n,q}} \le \|f\|_{J,q} < \infty.
\end{aligned}$$

On the other hand, as we mentioned in section 1, since $q<\infty$ then $L_{\infty,q}(X,\mu)=\{0\}$ and since $\int_0^\infty f^*(t)^q\frac{dt}{t}<\infty$, so we have $f=0$, $\mu-$almost every where on X. Thus $IL_{J,q}(X,\mu)=L_{\infty,q}(X,\mu)=\{0\}$. It follows that $IL_{J,q}(X,\mu)\subseteq L_{M_J,q}(X,\mu)$.

Now suppose that $q=\infty$ and $f\in IL_{J,q}(X,\mu)$. Then

$$\begin{aligned}\|f\|_{L_{M_J,\infty}} &= \sup_{t>0} t^{\frac{1}{M_J}}\, f^*(t)=\sup_{t>0}\left(\lim_n t^{\frac{1}{x_n}}.f^*(t)\right)\\ &\leq \sup_{t>0}\left(\lim_n \|f\|_{L_{x_n,\infty}}\right)=\|f\|_{L_{J,\infty}}<\infty,\end{aligned}$$

and so $f\in L_{M_J,\infty}(X,\mu)$. Thus we proved that $IL_{J,q}(X,\mu)\subseteq L_{M_J,q}(X,\mu)$, for each $0<q\leq\infty$. Using some similar arguments, one can obtain that $IL_{J,q}(X,\mu)\subseteq L_{m_J,q}(X,\mu)$. Consequently the proof is complete. □

The following proposition is obtained immediately from Propositions 2.2, 2.3 and 2.4.

Proposition 2.5. *Let (X,μ) be a measure space, $0<q\leq\infty$ and $J\subseteq(0,\infty)$ such that $m_J>0$ and $M_J<\infty$. Then*

$$\begin{aligned}IL_{J,q}(X,\mu) &= IL_{(m_J,M_J),q}(X,\mu)=IL_{[m_J,M_J),q}(X,\mu)\\ &= IL_{(m_J,M_J],q}(X,\mu)=IL_{[m_J,M_J],q}(X,\mu)\\ &= L_{m_J,q}(X,\mu)\cap L_{M_J,q}(X,\mu).\end{aligned}$$

Furthermore, for each $f\in IL_{J,q}(X,\mu)$ and $p\in J$,

$$\|f\|_{L_{p,q}}\leq 2^{1/q}\max\{\|f\|_{L_{m_J,q}},\|f\|_{L_{M_J,q}}\}$$

Theorem 2.6. *Let (X,μ) be a measure space and $J,Q\subseteq(0,\infty)$ such that $m_J,m_Q>0$. Then*

$$IL_{J,Q}(X,\mu)=L_{m_J,m_Q}(X,\mu)\cap L_{M_J,m_Q}(X,\mu). \tag{2.2}$$

Moreover for each $f\in IL_{J,Q}(X,\mu)$

$$\begin{aligned}\max\{\|f\|_{L_{m_J,m_Q}},\|f\|_{L_{M_J,m_Q}}\} &\leq \sup_{p\in J,q\in Q}\|f\|_{L_{p,q}}\\ &\leq K\max\{\|f\|_{L_{m_J,m_Q}},\|f\|_{L_{M_J,m_Q}}\},\end{aligned}$$

for some positive constant $K>0$.

Proof. Let $f\in IL_{J,Q}(X,\mu)$. Then by proposition 2.5, we have

$$f\in L_{m_J,q}(X,\mu)\cap L_{M_J,q}(X,\mu),$$

for each $q\in Q$, and so

$$f\in\left(\cap_{q\in Q}L_{m_J,q}(X,\mu)\right)\cap\left(\cap_{q\in Q}L_{M_J,q}(X,\mu)\right).$$

Thus [2, Theorem 12] implies that $f\in L_{m_J,m_Q}(X,\mu)\cap L_{M_J,m_Q}(X,\mu)$. Also by Fatou's lemma, one can readily obtain that

$$\|f\|_{L_{m_J,m_Q}}\leq\sup_{p\in J,q\in Q}\|f\|_{L_{p,q}}<\infty$$

and also

$$\|f\|_{L_{M_J,m_Q}}\leq\sup_{p\in J,q\in Q}\|f\|_{L_{p,q}}<\infty.$$

For the converse, note that by Proposition 2.2 and [3, Proposition 1.4.10], we have

$$\begin{aligned}L_{m_J,m_Q}(X,\mu)\cap L_{M_J,m_Q}(X,\mu) &= \bigcap_{m_J\leq r\leq M_J} L_{r,m_Q}(X,\mu)\\ &\subseteq \bigcap_{m_J\leq r\leq M_J,m_Q\leq t\leq M_Q} L_{r,t}(X,\mu).\end{aligned}$$

It follows that

$$L_{m_J,m_Q}(X,\mu)\cap L_{M_J,m_Q}(X,\mu)\subseteq \bigcap_{p\in J, q\in Q} L_{p,q}(X,\mu).$$

By Proposition 2.5 and [2, Theorem 12], for each $f\in L_{m_J,m_Q}(X,\mu)\cap L_{M_J,m_Q}(X,\mu)$ we have

$$\begin{aligned}
\max\{\|f\|_{L_{m_J,m_Q}},\|f\|_{L_{M_J,m_Q}}\} &\le \sup_{m_J\le p\le M_J, m_Q\le q\le M_Q}\|f\|_{L_{p,q}}\\
&\le \sup_{m_J\le p\le M_J}\left[\max\{1,(\frac{m_Q}{p})^{\frac{1}{m_Q}}\}\ \|f\|_{L_{p,m_Q}}\right]\\
&\le 2^{1/m_Q}\max\{1,(\frac{m_Q}{m_J})^{\frac{1}{m_Q}}\}\ \max\{\|f\|_{L_{m_J,m_Q}},\|f\|_{L_{M_J,m_Q}}\}
\end{aligned}$$

and so the inequality is provided by choosing

$$K=2^{1/m_Q}\max\{1,(\frac{m_Q}{m_J})^{\frac{1}{m_Q}}\}.$$

Moreover $f\in IL_{J,Q}(X,\mu)$ and the equality (2.2) is satisfied. □

Proposition 2.7. *Let (X,μ) be a measure space and $0<p\le\infty$. Then $fg\in L_{p,\infty}(X,\mu)$, for each $f\in L^\infty(X,\mu)$ and $g\in L_{p,\infty}(X,\mu)$.*

Proof. By [3, Proposition 1.4.5] parts (7) and (15), we have

$$\begin{aligned}
\|f\,g\|_{p,\infty} &= \sup_{t>0}\Big(t^{\frac{1}{p}}\ (f\,g)^*(t)\Big)\le\sup_{t>0}\Big(t^{\frac{1}{p}}f^*(\frac{t}{2})\ g^*(\frac{t}{2})\Big)\\
&= \sup_{t>0}\Big((2t)^{\frac{1}{p}}\ f^*(t)\ g^*(t)\Big)\le 2^{\frac{1}{p}}\ \|g\|_{p,\infty}\ \|f\|_\infty\\
&< \infty.
\end{aligned}$$

It follows that $fg\in L_{p,\infty}(X,\mu)$. □

Proposition 2.8. *Let (X,μ) be a measure space, $0<p\le\infty$ and $J,Q\subseteq(0,\infty)$ such that $m_J>0$, $m_Q>0$ and $m_Q\in Q$. Then $IL_{J,Q}(X,\mu)=A\cap B$, where*

$$A=\{f\in\bigcap_{p\in J,q\in Q}L_{p,q}(X,\mu),\ M_q=\sup_{p\in J}\|f\|_{L_{p,q}}<\infty,\ \forall q\in Q\}$$

and

$$B=\{f\in\bigcap_{p\in J,q\in Q}L_{p,q}(X,\mu),\ M_p=\sup_{q\in Q}\|f\|_{L_{p,q}}<\infty,\ \forall p\in J\}.$$

Proof. It is clear that $IL_{J,Q}(X,\mu)\subseteq A\cap B$. For the converse assume that $f\in A\cap B$. By [2, Theorem 12] implies that for each $p\in J$

$$\sup_{q\in Q}\|f\|_{L_{p,q}}\le\max\{1,(\frac{m_Q}{p})^{\frac{1}{m_Q}}\}\|f\|_{L_{p,m_Q}}$$

and so

$$\begin{aligned}
\sup_{p\in J,\ q\in Q}\|f\|_{L_{p,q}} &\le \max\{1,(\frac{m_Q}{m_J})^{\frac{1}{m_Q}}\}\sup_{p\in J}\|f\|_{L_{p,m_Q}}\\
&= \max\{1,(\frac{m_Q}{m_J})^{\frac{1}{m_Q}}\}\ M_{m_Q}\\
&< \infty.
\end{aligned}$$

It follows that $f\in IL_{J,Q}(X,\mu)$. □

In the sequel, we investigate some previous results, for the special Lorentz space $\ell_{p,q}\{E\}$, introduced in [4]. In the further discussions, E stands for a Banach space. Also K is the real or complex field and I is the set of positive integers. We first provide the required preliminaries, which follow from [4].

Definition 2.9. For $1 \leq p \leq \infty$, $1 \leq q < \infty$ or $1 \leq p < \infty$, $q = \infty$ let $\ell_{p,q}\{E\}$ be the space of all E-valued zero sequences $\{x_i\}$ such that

$$\|\{x_i\}\|_{p,q} = \begin{cases} \left(\sum_{i=1}^{\infty} i^{q/p-1} \|x_{\phi(i)}\|^{\ q}\right)^{\frac{1}{q}} & \text{for } 1 \leq p \leq \infty,\ 1 \leq q < \infty \\ \sup_i i^{\frac{1}{p}} \|x_{\phi(i)}\| & \text{for } 1 \leq p < \infty,\ q = \infty. \end{cases}$$

is finite, where $\{\|x_{\phi(i)}\|\}$ is the non-increasing rearrangement of $\{\|x_i\|\}$. If $E = K$, then $\ell_{p,q}\{K\}$ is denoted by $\ell_{p,q}$.

In particular, $\ell_{p,p}\{E\}$ coincides with $\ell_p\{E\}$ and $\|.\|_{p,p} = \|.\|_p$; see [5].

The following result will be used in the final result of this paper. It is in fact [4, Proposition 2].

Proposition 2.10. *Let E be a Banach space.*

(i) *If $1 \leq p < \infty, 1 \leq q < q_1 \leq \infty$, then $\ell_{p,q}\{E\} \subseteq \ell_{p,q_1}\{E\}$ and for every $\{x_i\} \in \ell_{p,q}\{E\}$*

$$\|\{x_i\}\|_{p,q_1} \leq \left(\frac{q}{p}\right)^{\frac{1}{q} - \frac{1}{q_1}} \|\{x_i\}\|_{p,q},$$

for $p < q$ and

$$\|\{x_i\}\|_{p,q_1} \leq \|\{x_i\}\|_{p,q},$$

for $p \geq q$. In fact

$$\|\{x_i\}\|_{p,q_1} \leq \max\{1, \frac{q}{p}\} \|\{x_i\}\|_{p,q}.$$

(ii) *Let eather $1 \leq p < p_1 \leq \infty$, $1 \leq q < \infty$ or $1 \leq p < p_1 < \infty$, $q = \infty$. Then*

$$\ell_{p,q}\{E\} \subseteq \ell_{p_1,q}\{E\}$$

and for every $\{x_i\} \in \ell_{p,q}\{E\}$

$$\|\{x_i\}\|_{p_1,q} \leq \|\{x_i\}\|_{p,q}$$

Now for $J, Q \subseteq [1, \infty)$ let

$$IL_{J,Q} = \{\{x_i\} \in \textstyle\bigcap_{p \in J, q \in Q} \ell_{p,q} : \ \ \|\{x_i\}\|_{J,Q} = \sup_{p \in J, q \in Q} \|\{x_i\}\|_{p,q} < \infty\}.$$

We finish this work with the following result, which determines the structure of $IL_{J,Q}$.

Proposition 2.11. *Let $J, Q \subseteq [1, \infty)$. Then $IL_{J,Q} = \ell_{m_J, m_Q}$.*

Proof. Some similar arguments to [2, Theorem 12] implies that $IL_{p,Q} = \ell_{p,m_Q}$. Indeed, by Proposition 2.10 for each $q \in Q$, $\ell_{p,m_Q} \subseteq \ell_{p,q}$. Also for each $\{x_i\} \in \ell_{p,m_Q}$,

$$\|\{x_i\}\|_{p,q} \leq \max\{1, \frac{m_Q}{p}\} \|\{x_i\}\|_{p,m_Q}.$$

It follows that $\{x_i\} \in IL_{p,Q}$ and

$$\|\{x_i\}\|_{p,Q} \leq \max\{1, \frac{m_Q}{p}\} \|\{x_i\}\|_{p,m_Q}.$$

Thus $\ell_{p,m_Q} \subseteq IL_{p,Q}$. The reverse of this inclusion is clear whenever $m_Q \in Q$. Now let $m_Q \notin Q$. Thus there is a sequence $(y_n)_{n\in\mathbb{N}}$ in Q, converging to m_Q. For each $\{x_i\} \in IL_{p,Q}$, Fatou's lemma implies that

$$\begin{aligned}
\|\{x_i\}\|_{p,m_Q}^{m_Q} &= \sum_{i=1}^{\infty} i^{\frac{m_Q}{p}-1}\|x_{\Phi(i)}\|^{m_Q} \\
&= \sum_{i=1}^{\infty} \liminf_n \left(i^{\frac{y_n}{p}-1}\|x_{\Phi(i)}\|^{y_n}\right) \\
&\leq \liminf_n \sum_{i=1}^{\infty} \left(i^{\frac{y_n}{p}-1}\|x_{\Phi(i)}\|^{y_n}\right) \\
&= \liminf_n \|\{x_i\}\|_{p,y_n}^{y_n} \leq \liminf_n \|\{x_i\}\|_{p,Q}^{y_n} \\
&= \|\{x_i\}\|_{p,Q}^{m_Q},
\end{aligned}$$

which implies $\{x_i\} \in \ell_{p,m_Q}$. Consequently $IL_{p,Q} = \ell_{p,m_Q}$. In the sequel, we show that $IL_{J,q} \subseteq \ell_{m_J,q}$, for each $q \in Q$. Suppose that (y_n) be a sequences in J such that $\lim_n x_n = m_J$ and $\{x_i\} \in IL_{J,q}$ Then by Fatou's lemma, we have

$$\begin{aligned}
\|\{x_i\}\|_{m_J,q}^{q} &= \sum_{i=1}^{\infty} \left(i^{\frac{q}{m_J}-1}\|x_{\Phi(i)}\|^{q}\right) \\
&= \sum_{i=1}^{\infty} \liminf_n \left(i^{\frac{q}{y_n}-1}\|x_{\Phi(i)}\|^{q}\right) \\
&\leq \liminf_n \sum_{i=1}^{\infty} \left(i^{\frac{q}{y_n}-1}\|x_{\Phi(i)}\|^{q}\right) \\
&= \liminf_n \|\{x_i\}\|_{y_n,q}^{q} \\
&\leq \|\{x_i\}\|_{J,q}^{q} \\
&< \infty.
\end{aligned}$$

Hence $IL_{J,q} \subseteq \ell_{m_J,q}$. Now suppose that $\{x_i\} \in IL_{J,Q}$. Then for each $q \in Q$, $\{x_i\} \in IL_{J,q}$ and so $\{x_i\} \in \ell_{m_J,q}$. On the other hand by the above inequalities, for each $1 \leq q < \infty$, we have $\|\{x_i\}\|_{m_J,q} \leq \|\{x_i\}\|_{J,q}$. So $\{x_i\} \in IL_{m_J,Q} \subseteq \ell_{m_J,m_Q}$, which implies $IL_{J,Q} \subseteq \ell_{m_J,m_Q}$. Also by Proposition 2.10, for each $p \geq m_J$ and $q \geq m_Q$ we have $\ell_{m_J,m_Q} \subseteq \ell_{m_J,q} \subseteq \ell_{p,q}$. Consequently

$$\ell_{m_J,m_Q} \subseteq \bigcap_{p\in J, q\in Q} \ell_{p,q}.$$

Moreover for each $\{x_i\} \in \ell_{p,q}$,

$$\sup_{p\in J,q\in Q} \|\{x_i\}\|_{p,q} \leq \sup_{q\in Q} \|\{x_i\}\|_{m_J,q} \leq \max\{1, \frac{m_Q}{m_J}\}\|\{x_i\}\|_{m_J,m_Q}.$$

It follows that

$$\ell_{m_J,m_Q} \subseteq IL_{J,Q} \subseteq \ell_{m_J,m_Q}.$$

Therefore $IL_{J,Q} = \ell_{m_J,m_Q}$, as claimed. $\square$

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